# On pionic hydrogen. Quantum field theoretic, relativistic covariant and model-independent approach

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**Abstract.** We consider pionic hydrogen $A_{\pi p}$, the bound $\pi^- p$ state. Within the quantum field theoretic and relativistic covariant approach we calculate the shift and width of the energy level of the ground state of pionic hydrogen caused by strong low-energy interactions treated perturbatively. The generalization of the Deser-Goldberger-Baumann-Thirring (DGBT) formulas (S. Deser, M.L. Goldberger, K. Baumann, W. Thirring, Phys. Rev. **96**, 774 (1954)) is given. The generalized DGBT formulas for the energy level displacement of the ground state of pionic hydrogen contain the non-perturbative and model-independent correction of about 1%, caused by the relativistic covariant smearing of the wave function of the ground state around the origin. This non-perturbative correction is very important for the precise extraction of the $S$-wave scattering lengths of the $\pi N$ scattering from the experimental data on the energy level displacements in pionic hydrogen by the PSI Collaboration. In addition, the shift of the energy level of the ground state of pionic hydrogen is improved by the second-order correction of strong low-energy interactions which is about 0.1%. This testifies the applicability of the perturbative treatment of strong low-energy interactions to the analysis of pionic hydrogen. We show that the width of the energy level of the ground state of pionic hydrogen is valid to all orders of the perturbation theory in strong low-energy interactions.

**PACS.** 11.10.Ef Lagrangian and Hamiltonian approach – 11.55.Ds Exact $S$ matrices – 13.75.Gx Pion-baryon interactions – 21.10.-k Properties of nuclei; nuclear energy levels – 36.10.-k Exotic atoms and molecules (containing mesons, muons, and other unusual particles)

## 1 Introduction

Pionic hydrogen $A_{\pi p}$ is an analogy of the hydrogen with an electron replaced by the $\pi^-$-meson. The existence of pionic hydrogen is fully due to the Coulomb forces [1–3]. The Bohr radius of pionic hydrogen is equal to

$$a_{\mathrm{B}} = \frac{1}{\mu\alpha} = \frac{1}{\alpha}\left(\frac{1}{m_{\pi^-}} + \frac{1}{m_p}\right) = 222.664\,\mathrm{fm}, \qquad (1.1)$$

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where $\mu = m_{\pi^-} m_p/(m_{\pi^-} + m_p) = 121.497\,\mathrm{MeV}$ is the reduced mass of the $\pi^- p$ system, calculated at $m_{\pi^-} = 139.570\,\mathrm{MeV}$ and $m_p = 938.272\,\mathrm{MeV}$ [4], and $\alpha = e^2/\hbar c = 1/137.036$ is the fine-structure constant defined [4]. Below we use the units $\hbar = c = 1$, then $\alpha = e^2 = 1/137.036$.

Since the Bohr radius of pionic hydrogen is much greater than the radius of strong interactions $R_{\mathrm{str}} \sim 1/m_{\pi^-} = 1.414\,\mathrm{fm}$, the strong low-energy interactions can be taken into account perturbatively [1]. Since the mass of the state $\pi^0 n$ is less than the mass of pionic hydrogen, $m_{\pi^0} = 134.977\,\mathrm{MeV}$ and $m_n = 939.565\,\mathrm{MeV}$ [4], the mesoatom $A_{\pi p}$ is unstable under the decay $A_{\pi p} \to \pi^0 + n$ [1] caused by strong low-energy interactions. This decay goes through the intermediate $\pi N$ scattering, *i.e.* $A_{\pi p} \to \pi^- + p \to \pi^0 + n$, the s-channel amplitude of which is determined by two states with isotopic spin $I = 1/2$ and $I = 3/2$. Near threshold the amplitudes of the $\pi N$ scattering with isotopic spin $I = 1/2$ and $I = 3/2$ are

defined by the $S$-wave scattering lengths $a_0^{1/2}$ and $a_0^{3/2}$, respectively. The relative momenta $p$ of the $\pi^- p$ system in the ground state of pionic hydrogen are of order $p \sim 1/a_{\rm B} = 0.887\,{\rm MeV}$ and smaller compared with the reduced mass of the $\pi N$ system $\mu = 121.497\,{\rm MeV}$; therefore the low-energy limit for the calculation of the amplitude of the $\pi N$ scattering is well defined. As a result, the shift and width of the energy level of the ground state of pionic hydrogen should be expressed in terms of the $S$-wave scattering lengths of the low-energy $\pi N$ scattering.

As has been found by Deser, Goldberger, Baumann and Thirring [1], due to strong low-energy interactions the energy level of the ground state of pionic hydrogen has the following shift and width:

$$
\begin{aligned}
\epsilon_{1s} &= -\frac{2\pi}{3}\frac{1}{\mu}\left(2a_0^{1/2}+a_0^{3/2}\right)|\Psi_{1s}(0)|^2, \\
\Gamma_{1s} &= \frac{8\pi}{9}\frac{p^*}{\mu}\left(a_0^{1/2}-a_0^{3/2}\right)^2|\Psi_{1s}(0)|^2,
\end{aligned}
\tag{1.2}
$$

where the relative momentum $p^*$ is equal to

$$
\begin{aligned}
p^* &= \frac{m_p + m_{\pi^-}}{2} \\
&\times \sqrt{\left[1-\left(\frac{m_n+m_{\pi^0}}{m_p+m_{\pi^-}}\right)^2\right]\left[1-\left(\frac{m_n-m_{\pi^0}}{m_p+m_{\pi^-}}\right)^2\right]} = \\
&\quad 28.040\,{\rm MeV}
\end{aligned}
\tag{1.3}
$$

and $\Psi_{1s}(0) = 1/\sqrt{\pi a_{\rm B}^3}$ is the wave function of the pionic hydrogen in the ground state

$$
\Psi_{1s}(\vec{r}) = \frac{1}{\sqrt{\pi a_{\rm B}^3}}\,e^{-r/a_{\rm B}}
\tag{1.4}
$$

at the origin $r = 0$. We emphasize that the width $\Gamma_{1s}$ should be related to the imaginary part of the energy level shift $E_{1s}$ by the relation $\Gamma_{1s} = -2\,\mathcal{I}m E_{1s}$ [1].

The DGBT formulas (1.2) can be transcribed into an equivalent form given by Deser, Goldberger, Baumann and Thirring [1]

$$
\begin{aligned}
-\frac{\epsilon_{1s}}{E_{1s}} &= -\frac{4}{3}\frac{1}{a_{\rm B}}\left(2a_0^{1/2}+a_0^{3/2}\right), \\
-\frac{\Gamma_{1s}}{E_{1s}} &= \frac{16}{9}\frac{p^*}{a_{\rm B}}\left(a_0^{1/2}-a_0^{3/2}\right)^2,
\end{aligned}
\tag{1.5}
$$

where $E_{1s} = -\alpha/2a_{\rm B}$ is the binding energy of the ground state of pionic hydrogen.

All attempts of the generalization of the relations (1.5) have been undertaken within the quantum-mechanical potential non-relativistic approach [2].

The accuracy of the modern level of experimental analysis of the parameters of pionic hydrogen reached by the PSI Collaboration is about 0.2% for the shift and 1% for the width of the energy level of the ground state of pionic hydrogen [5]. Since the derivation of the relations (1.2)

has been carried out within the potential non-relativistic approach, the modern level of experimental accuracy demands the derivation of the shift and width of the energy level of the ground state of pionic hydrogen in a fully quantum field theoretic and relativistic covariant approach.

We would like to accentuate that a quantum field theoretical analysis of the DGBT formulas has been recently carried out by Lyubovitskij and Rusetsky [6] and Lyubovitskij *et al.* [7]. They have performed a consistent analysis of QCD isospin-breaking and electromagnetic corrections to the shift of the energy level described by (1.2). Numerically, the QCD isospin-breaking and electromagnetic corrections calculated in [6,7] are compared well with results obtained within the potential model approach [8].

The paper is organized as follows: In sect. 2 we determine the wave function of the ground state of pionic hydrogen within the quantum field theoretic and relativistic covariant approach. In sect. 3 we give general formulas for the energy level shift $\epsilon_{1s}$ and the width $\Gamma_{1s}$ of the ground state of pionic hydrogen within the quantum field theoretic and relativistic covariant approach. In sect. 4 we calculate $\epsilon_{1s}^{(1)}$, the shift of the energy level of the ground state of pionic hydrogen to the first order of the perturbation theory. The obtained result we represent as $\epsilon_{1s}^{(1)} = \epsilon_{1s}^{(0)}(1+\delta_{1s}^{(\rm s)})$, where $\epsilon_{1s}^{(0)}$ is defined by (1.5). The correction $\delta_{1s}^{(\rm s)} = -9.69\times 10^{-3}$ is caused by the relativistic smearing of the wave function of the ground state of pionic hydrogen around the origin $r = 0$. In sect. 5 we calculate $\epsilon_{1s}^{(2)} = \epsilon_{1s}^{(0)}\delta_{1s}^{(2)}$, the shift of the energy level to the second order of the perturbation theory in strong low-energy interactions, and the width $\Gamma_{1s} = \Gamma_{1s}^{(0)}(1+\delta_{1s}^{(\rm s)})$, where $\Gamma_{1s}^{(0)}$ is given by (1.5), of the energy level of the ground state of pionic hydrogen. In sect. 6 we discuss a removal of ultra-violet divergences of $\epsilon_{1s}^{(2)}$ by renormalization of the reduced mass of the $\pi^- p$ system. In sect. 7 we summarize the contributions to the energy level displacement of the ground state of pionic hydrogen and give the generalization of the DGBT formulas: i) $\epsilon_{1s} = \epsilon_{1s}^{(0)}\left(1+\delta_{1s}^{(\rm s)}+\delta_{1s}^{(2)}\right)$ and ii) $\Gamma_{1s} = \Gamma_{1s}^{(0)}\left(1+\delta_{1s}^{(\rm s)}\right)$. In sect. 8, using the experimental data on the $S$-wave scattering lengths of the $\pi N$ scattering obtained by the PSI Collaboration [9], we estimate the ratio $\delta_{1s}^{(2)} = \epsilon_{1s}^{(2)}/\epsilon_{1s}^{(1)} = (0.111\pm 0.006)\%$. This testifies the applicability of the perturbative treatment of strong low-energy interactions to the analysis of the energy level displacements of pionic hydrogen. We show also that the width of the energy level $\Gamma_{1s} = \Gamma_{1s}^{(0)}\left(1+\delta_{1s}^{(\rm s)}\right)$ is valid to all orders in the perturbation theory. The former is due to the Adler consistency condition [10,11]. In the conclusion we discuss the obtained results. In appendix A we calculate the momentum integral (4.6) defining the energy level displacement of the ground state of pionic hydrogen within the quantum field theoretic and relativistic covariant approach. In appendix B we have relegated the lengthy intermediate calculations of $\epsilon_{1s}^{(2)}$ and $\Gamma_{1s}$.

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[1] Recall that in ref. [1] there has been calculated the semi-width defined by $\Gamma_{1s} = -\mathcal{I}m E_{1s}$.

## 2 Ground-state wave function of pionic hydrogen

The wave function of pionic hydrogen in the ground state we define as [12,13]

$$|A_{\pi p}^{(1s)}(\vec{P}, \sigma_p)\rangle = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k_{\pi^-}}{\sqrt{2E_{\pi^-}(\vec{k}_{\pi^-})}}$$

$$\times \frac{\mathrm{d}^3 k_p}{\sqrt{2E_p(\vec{k}_p)}} \delta^{(3)}(\vec{P} - \vec{k}_{\pi^-} - \vec{k}_p)$$

$$\times \sqrt{2E_A^{(1s)}(\vec{k}_{\pi^-} + \vec{k}_p)} \, \Phi_{1s}(\vec{k}_{\pi^-}) |\pi^-(\vec{k}_{\pi^-}) p(\vec{k}_p, \sigma_p)\rangle, \quad (2.1)$$

where $E_A^{(1s)}(\vec{P}) = \sqrt{M_A^{(n)\,2} + \vec{P}^{\,2}}$ and $\vec{P}$ are the total energy and the momentum of pionic hydrogen, respectively; $M_A^{(1s)} = m_p + m_{\pi^-} + E_{1s}$ and $E_{1s}$ are the mass and the binding energy of pionic hydrogen in the ground bound state, respectively; $\sigma_p$ is the polarization. Then, $\Phi_{1s}(\vec{k}_{\pi^-})$ is the wave function of the ground state in the momentum representation. It is normalized as

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, |\Phi_{1s}(\vec{k})|^2 = 1. \quad (2.2)$$

The wave function $|\pi^-(\vec{k}_{\pi^-}) p(\vec{k}_p, \sigma_p)\rangle$ we define as [12,13]

$$|\pi^-(\vec{k}_{\pi^-}) p(\vec{k}_p, \sigma_p)\rangle = c_{\pi^-}^\dagger(\vec{k}_{\pi^-}) a_p^\dagger(\vec{k}_p, \sigma_p)|0\rangle, \quad (2.3)$$

where $c_{\pi^-}^\dagger(\vec{k}_{\pi^-})$ and $a_p^\dagger(\vec{k}_p, \sigma_p)$ are operators of creation of the $\pi^-$-meson with momentum $\vec{k}_{\pi^-}$ and the proton with momentum $\vec{k}_p$ and polarization $\sigma_p = \pm 1/2$. They satisfy standard relativistic covariant commutation and anti-commutation relations

$$[c_{\pi^-}(\vec{k}\,'_{\pi^-}), c_{\pi^-}^\dagger(\vec{k}_{\pi^-})] = (2\pi)^3 \, 2E_{\pi^-}(\vec{k}_{\pi^-}) \, \delta^{(3)}(\vec{k}\,'_{\pi^-} - \vec{k}_{\pi^-}),$$

$$[c_{\pi^-}(\vec{k}\,'_{\pi^-}), c_{\pi^-}(\vec{k}_{\pi^-})] = [c_{\pi^-}^\dagger(\vec{k}\,'_{\pi^-}), c_{\pi^-}^\dagger(\vec{k}_{\pi^-})] = 0,$$

$$\{a_p(\vec{k}\,'_p, \sigma\,'_p), a_p^\dagger(\vec{k}_p, \sigma_p)\} =$$
$$(2\pi)^3 \, 2E_p(\vec{k}_p) \, \delta^{(3)}(\vec{k}\,'_p - \vec{k}_p) \, \delta_{\sigma\,'_p \sigma_p},$$

$$\{a_p(\vec{k}\,'_p, \sigma\,'_p), a_p(\vec{k}_p, \sigma_p)\} = \{a_p^\dagger(\vec{k}\,'_p, \sigma\,'_p), a_p^\dagger(\vec{k}_p, \sigma_p)\} = 0. \quad (2.4)$$

The wave function (2.1) is normalized by

$$\langle A_{\pi p}^{(1s)}(\vec{P}', \sigma\,'_p) | A_{\pi p}^{(1s)}(\vec{P}, \sigma_p)\rangle =$$
$$(2\pi)^3 \, 2E_A^{(1s)}(\vec{P}) \, \delta^{(3)}(\vec{P}' - \vec{P}) \, \delta_{\sigma\,'_p \sigma_p}$$
$$\times \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, |\Phi_{1s}(\vec{k})|^2 =$$
$$(2\pi)^3 \, 2E_A^{(1s)}(\vec{P}) \, \delta^{(3)}(\vec{P}' - \vec{P}) \, \delta_{\sigma\,'_p \sigma_p}. \quad (2.5)$$

This is a relativistic covariant normalization of the wave function.

We will apply the wave function (2.1) to the calculation of the energy level displacement of the ground state of pionic hydrogen within the quantum field theoretic and the relativistic covariant approach.

## 3 Energy level displacement of the ground state. Quantum field theoretic approach

The quantum field theoretic description of strong low-energy interactions can be carried out by the *effective* Lagrangian $\mathcal{L}_{\mathrm{str}}(x)$. For the quantum field theoretic and model-independent derivation of the DGBT formulas (1.2) we will not specify $\mathcal{L}_{\mathrm{str}}(x)$ in terms of interpolating fields of the coupled mesons and baryons. We would like to emphasize that $\mathcal{L}_{\mathrm{str}}(x)$ is a total *effective* Lagrangian accounting for all strong interactions. In other words this *effective* Lagrangian defines the $\mathbb{T}_{\mathrm{str}}$-matrix of strong interactions

$$\mathbb{T}_{\mathrm{str}} = \int \mathrm{d}^4 x \, \mathcal{L}_{\mathrm{str}}(x) \quad (3.1)$$

obeying the unitary condition [12,14]

$$\mathbb{T}_{\mathrm{str}} - \mathbb{T}_{\mathrm{str}}^\dagger = i \mathbb{T}_{\mathrm{str}}^\dagger \mathbb{T}_{\mathrm{str}}. \quad (3.2)$$

This means that the matrix element of the *effective* Lagrangian $\mathcal{L}_{\mathrm{str}}(0)$ between the initial state $|N^i(q_i, \sigma_i)\pi^a(p_a)\rangle$ and the final state $|N^j(q_j, \sigma_j)\pi^b(p_b)\rangle$ defines the total amplitude of the $\pi N$ scattering [15,16]

$$\langle \pi^b(p_b) N^j(q_j, \sigma_j) | \mathcal{L}_{\mathrm{str}}(0) | N^i(q_i, \sigma_i) \pi^a(p_a)\rangle =$$
$$\delta^{ba} \delta^{ji} \frac{1}{3} (T^{1/2} + 2T^{3/2})_{\sigma_j \sigma_i}$$
$$- i \, \epsilon^{bac} \tau^{cji} \frac{1}{3} (T^{1/2} - T^{3/2})_{\sigma_j \sigma_i}, \quad (3.3)$$

where $T_{\sigma_j \sigma_i}^{1/2}$ and $T_{\sigma_j \sigma_i}^{3/2}$ are amplitudes of the $\pi N$ scattering with isotopic spin $I = 1/2$ and $I = 3/2$, respectively, $\epsilon^{bac}$ is the anti-symmetric unit tensor, $a(b) = 1, 2, 3$ and $i(j) = 1, 2$ are isotopic indices of pions and nucleons, $\tau^c$ ($c = 1, 2, 3$) are $2 \times 2$ Pauli matrices. From (3.3) for given channels of the $\pi N$ scattering we get [16]

$$\langle \pi^+ p | \mathcal{L}_{\mathrm{str}}(0) | p \, \pi^+\rangle = \langle \pi^- n | \mathcal{L}_{\mathrm{str}}(0) | n \, \pi^-\rangle = T^{3/2},$$
$$\langle \pi^- p | \mathcal{L}_{\mathrm{str}}(0) | p \, \pi^-\rangle = \langle \pi^+ n | \mathcal{L}_{\mathrm{str}}(0) | n \, \pi^+\rangle =$$
$$\frac{1}{3} (2T^{1/2} + T^{3/2}),$$
$$\langle \pi^0 n | \mathcal{L}_{\mathrm{str}}(0) | p \, \pi^-\rangle = \langle \pi^+ n | \mathcal{L}_{\mathrm{str}}(0) | p \, \pi^0\rangle =$$
$$\frac{\sqrt{2}}{3} (T^{3/2} - T^{1/2}),$$
$$\langle \pi^0 n | \mathcal{L}_{\mathrm{str}}(0) | n \, \pi^0\rangle = \frac{1}{3} (T^{1/2} + 2T^{3/2}). \quad (3.4)$$

At threshold the amplitudes $T^{1/2}$ and $T^{3/2}$ are proportional to the $S$-wave scattering lengths $a_0^{1/2}$ and $a_0^{3/2}$.

Since for the calculation of the shift and width of the energy of the pionic hydrogen ground state the strong interactions can be treated as a perturbation, we define the $S$-matrix [13] as

$$S = 1 + i \, \mathbb{T} = \mathrm{T} \exp i \int \mathrm{d}^4 x \, \mathcal{L}_{\mathrm{str}}(x), \quad (3.5)$$

where T is a time-ordering operator.

We would like to emphasize that expanding the exponential in powers of $\mathcal{L}_{\text{str}}(x)$ and calculating the matrix elements of these operators for pionic-hydrogen ground state one encounters divergences which should be removed only by renormalization of the reduced mass of pionic hydrogen. This is in complete agreement with the calculation of the Lamb shift of hydrogen [17] (see also [18]).

It is important to notice that no hadronic loops should appear for the calculation of matrix elements of the expansion in powers of $\mathcal{L}_{\text{str}}(x)$. The perturbation theory with respect to $\mathcal{L}_{\text{str}}(x)$ will be developed in the tree-approximation. Therefore, the parameters of the low-energy $\pi N$ scattering, such as the $S$-wave scattering lengths $a_0^{1/2}$ and $a_0^{3/2}$ appearing in our expressions, are unrenormalizable. They define observable $S$-wave scattering lengths.

For the derivation of the DGBT formulas (1.2) it suffices to expand the exponential in powers of $\mathcal{L}_{\text{str}}(x)$ up to the second order inclusively:

$$
\begin{aligned}
S = {} & 1 + i\,\mathbb{T} = 1 + i \int \mathrm{d}^4 x_1\, \mathcal{L}_{\text{str}}(x_1) \\
& - \frac{1}{2} \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2\, \mathrm{T}(\mathcal{L}_{\text{str}}(x_1)\mathcal{L}_{\text{str}}(x_2)) + \dots \quad (3.6)
\end{aligned}
$$

The shift and width of the energy level of the ground state of the pionic hydrogen should be defined by the matrix element

$$
\begin{aligned}
& \langle A_{\pi p}^{(1s)}(\vec{P}\,',\sigma\,')|\mathbb{T}|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle = \int \mathrm{d}^4 x_1 \\
& \times \langle A_{\pi p}^{(1s)}(\vec{P}\,',\sigma\,')|\mathcal{L}_{\text{str}}(x_1)|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle + \frac{i}{2}\int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \\
& \times \langle A_{\pi p}^{(1s)}(\vec{P}\,',\sigma\,')|\mathrm{T}(\mathcal{L}_{\text{str}}(x_1)\mathcal{L}_{\text{str}}(x_2))|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle = \\
& (2\pi)^4 \delta^{(4)}(P'-P)\Big[\langle A_{\pi p}^{(1s)}(\vec{P}\,',\sigma\,')|\mathcal{L}_{\text{str}}(0)|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle \\
& + \frac{i}{2}\int \mathrm{d}^4 x\, \langle A_{\pi p}^{(1s)}(\vec{P}\,',\sigma\,')|\mathrm{T}(\mathcal{L}_{\text{str}}(x)\mathcal{L}_{\text{str}}(0))|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle\Big],
\end{aligned}
$$
$$(3.7)$$

where $|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle$ is the wave function of the $A_{\pi p}$ mesoatom in the ground bound state with momentum $\vec{P}$ and polarization $\sigma_p$.

Setting $\vec{P}\,' = \vec{P}$ and $\sigma\,' = \sigma_p$, we get

$$
\begin{aligned}
& \lim_{T,V\to\infty} \frac{1}{VT}\langle A_{\pi p}^{(1s)}(\vec{P},\sigma_p)|\mathbb{T}|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle = \\
& \langle A_{\pi p}^{(1s)}(\vec{P},\sigma_p)|\mathcal{L}_{\text{str}}(0)|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle \\
& + \frac{i}{2}\int \mathrm{d}^4 x\, \langle A_{\pi p}^{(1s)}(\vec{P},\sigma_p)|\mathrm{T}(\mathcal{L}_{\text{str}}(x)\mathcal{L}_{\text{str}}(0))|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle,
\end{aligned}
$$
$$(3.8)$$

where $TV$ is a 4-dimensional volume [12] defined by $(2\pi)^4\delta^{(4)}(0) = TV$.

According to [12], the energy level shift $\epsilon_{1s}$ and the partial width $\Gamma_{1s}$ can be defined by the matrix element (3.8)

at the rest frame of pionic hydrogen, where $\vec{P} = 0$, as

$$
\lim_{T,V\to\infty} \frac{\langle A_{\pi p}^{(1s)}(\vec{P},\sigma_p)|\mathbb{T}|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle}{2E_A^{(1s)}(\vec{P})VT}\Big|_{\vec{P}=0} = -\epsilon_{1s} + i\,\frac{\Gamma_{1s}}{2}.
$$
$$(3.9)$$

Formally, this is a general formula for the energy level displacement of the ground state of pionic hydrogen. It is valid to any order of the perturbation theory in strong low-energy interactions, where $\mathbb{T}$ is defined by (3.5).

The shift of the energy level $\epsilon_{1s}$, calculated to the second order of the perturbation theory in strong low-energy interactions, we determine as

$$
\epsilon_{1s} = \epsilon_{1s}^{(1)} + \epsilon_{1s}^{(2)}, \quad (3.10)
$$

where $\epsilon_{1s}^{(1)}$ and $\epsilon_{1s}^{(2)}$ are given by the first and second terms in (3.8), respectively. The partial width $\Gamma_{1s}$ is defined only by the contribution of the second term in (3.8).

# 4 Calculation of $\epsilon_{1s}^{(1)}$

The first term in (3.10) can be written as

$$
\begin{aligned}
& \langle A_{\pi p}^{(1s)}(\vec{P},\sigma_p)|\mathcal{L}_{\text{str}}(0)|A_{\pi p}^{(1s)}(\vec{P},\sigma_p)\rangle = 2E_A^{(1s)}(\vec{P})\frac{1}{(2\pi)^6} \\
& \times \int \frac{\mathrm{d}^3 k_{\pi^-}}{\sqrt{2E_{\pi^-}(\vec{k}_{\pi^-})}} \frac{\mathrm{d}^3 k_p}{\sqrt{2E_p(\vec{k}_p)}} \frac{\mathrm{d}^3 q_{\pi^-}}{\sqrt{2E_{\pi^-}(\vec{q}_{\pi^-})}} \frac{\mathrm{d}^3 q_p}{\sqrt{2E_p(\vec{q}_p)}} \\
& \times \delta^{(3)}(\vec{P}-\vec{k}_{\pi^-}-\vec{k}_p)\delta^{(3)}(\vec{P}-\vec{q}_{\pi^-}-\vec{q}_p)\Phi_{1s}^{\dagger}(\vec{k}_{\pi^-})\Phi_{1s}(\vec{q}_{\pi^-}) \\
& \times \langle \pi^-(\vec{k}_{\pi^-})p(\vec{k}_p,\sigma_p)|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q}_{\pi^-})p(\vec{q}_p,\sigma_p)\rangle. \quad (4.1)
\end{aligned}
$$

Setting $\vec{P} = 0$ and making necessary integrations we transcribe the r.h.s. of (4.1) into the form

$$
\begin{aligned}
& \langle A_{\pi p}^{(1s)}(0,\sigma_p)|\mathcal{L}_{\text{str}}(0)|A_{\pi p}^{(1s)}(0,\sigma_p)\rangle = 2M_A^{(1s)} \\
& \times \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\Phi_{1s}^{\dagger}(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\Phi_{1s}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}} \\
& \times \langle \pi^-(\vec{k})p(-\vec{k},\sigma_p)|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma_p)\rangle. \quad (4.2)
\end{aligned}
$$

The matrix element $\langle \pi^-(\vec{k})p(-\vec{k},\sigma_p)|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})$ $p(-\vec{q},\sigma_p)\rangle$ defines the amplitude of the elastic $\pi^- p$ scattering:

$$
\begin{aligned}
& \langle \pi^-(\vec{k})p(-\vec{k},\sigma_p)|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma_p)\rangle = \\
& \qquad \frac{1}{3}\,(2\,T^{1/2} + T^{3/2}). \quad (4.3)
\end{aligned}
$$

Since due to the wave functions $\Phi_{1s}^{\dagger}(\vec{k})$ and $\Phi_{1s}(\vec{q})$ the integrands are concentrated around momenta $q \sim k \sim 1/a_{\text{B}} = 0.887\,\text{MeV}$, which are smaller compared with the reduced mass of the $\pi^- p$ system $\mu = 121.497\,\text{MeV}$, the matrix element (4.3) can be calculated in the low-energy limit at $k = q = 0$. In the limit $k, q \to 0$ the amplitude

of the elastic $\pi^- p$ scattering can be expressed in terms of the $S$-wave scattering lengths $a_0^{1/2}$ and $a_0^{3/2}$ and reads

$$\lim_{k,q\to0}\langle\pi^-(\vec{k})p(-\vec{k},\sigma_p)|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma_p)\rangle =$$
$$\frac{8\pi}{3}\,(m_{\pi^-}+m_p)\,\left(2a_0^{1/2}+a_0^{3/2}\right). \qquad (4.4)$$

Substituting (4.4) in (4.2) we obtain

$$\langle A_{\pi p}^{(1s)}(0,\sigma_p)|\mathcal{L}_{\mathrm{str}}(0)|A_{\pi p}^{(1s)}(0,\sigma_p)\rangle =$$
$$2M_A^{(1s)}\,\frac{8\pi}{3}\,(m_{\pi^-}+m_p)\,\left(2a_0^{1/2}+a_0^{3/2}\right)$$
$$\times\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\Phi_{1s}^{\dagger}(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}}$$
$$\times\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\Phi_{1s}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}}. \qquad (4.5)$$

According to the definition (3.9), the energy level shift $\epsilon_{1s}^{(1)}$ is equal to

$$\epsilon_{1s}^{(1)} = -\frac{2\pi}{3}\frac{1}{\mu}\,\left(2a_0^{1/2}+a_0^{3/2}\right)$$
$$\times\left|\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{m_{\pi^-}m_p}{E_{\pi^-}(\vec{k})E_p(\vec{k})}}\,\Phi_{1s}(\vec{k})\right|^2. \qquad (4.6)$$

Formula (4.6) is a generalization of the DGBT formula due to the quantum field theoretic and relativistic covariant approach. The momentum integral in (4.6) is calculated in appendix A and the result reads (A.9)

$$\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{m_{\pi^-}m_p}{E_{\pi^-}(\vec{k})E_p(\vec{k})}}\,\Phi_{1s}(\vec{k}) = \Psi_{1s}(0)\left(1+\frac{1}{2}\delta_{1s}^{(\mathrm{s})}\right), \qquad (4.7)$$

where $\delta_{1s}^{(\mathrm{s})}$ is equal to

$$\delta_{1s}^{(\mathrm{s})} = -\alpha\,\frac{\mu}{m_{\pi^-}}\,\frac{8}{\sqrt{\pi}}\,\frac{\Gamma(3/4)}{\Gamma(1/4)} + O(\alpha^2) = -9.69\times10^{-3}. \qquad (4.8)$$

The parameter $\delta_{1s}^{(\mathrm{s})}$ defines the non-perturbative and model-independent correction caused by the quantum field theoretic and relativistic covariant approach. The index $s$ means the *smearing* of the wave function of the ground state of pionic hydrogen around the origin $r=0$ due to the relativistic factor $\sqrt{m_{\pi^-}m_p/E_{\pi^-}(\vec{k})E_p(\vec{k})}$.

Substituting (4.7) in (4.6) we represent the energy level shift $\epsilon_{1s}^{(1)}$ as

$$\epsilon_{1s}^{(1)} = -\frac{2\pi}{3}\frac{1}{\mu}\,\left(2a_0^{1/2}+a_0^{3/2}\right)|\Psi_{1s}(0)|^2\,\left(1+\delta_{1s}^{(\mathrm{s})}\right) \quad (4.9)$$

or in the equivalent form

$$-\frac{\epsilon_{1s}^{(1)}}{E_{1s}} = -\frac{4}{3}\frac{1}{a_{\mathrm{B}}}\,\left(2a_0^{1/2}+a_0^{3/2}\right)\left(1+\delta_{1s}^{(\mathrm{s})}\right). \qquad (4.10)$$

The non-perturbative correction $\delta_{1s}^{(\mathrm{s})}$ to the DGBT formula makes up 0.969%. It is important for the more precise extraction of the $S$-wave scattering lengths of the $\pi N$ scattering from the experimental data on the displacement of the energy level of the ground state of pionic hydrogen [5,9]. Recall that the precision of the experimental data on the shift of the energy level of the ground state of pionic hydrogen is about 0.2% [5] and 0.5% [9].

# 5 Calculation of $\epsilon_{1s}^{(2)}$ and $\Gamma_{1s}$

The energy level shift $\epsilon_{1s}^{(2)}$ and width $\Gamma_{1s}$ are defined by the second term in (3.8). The calculations of these terms are rather lengthy and we have relegated them to appendix B. For the energy level shift $\epsilon_{1s}^{(2)}$ we have obtained

$$\epsilon_{1s}^{(2)} = -\frac{2}{3}\frac{1}{\mu^2}\left[2(a_0^{1/2})^2+(a_0^{3/2})^2\right]$$
$$\times\int_0^{\infty}\frac{\mathrm{d}Q\,Q^2}{\sqrt{(m_{\pi^-}^2+Q^2)(m_p^2+Q^2)}}$$
$$\times\frac{m_{\pi^-}m_p}{\sqrt{m_{\pi^-}^2+Q^2}+\sqrt{m_p^2+Q^2}-m_{\pi^-}-m_p}$$
$$\times\left|\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{m_{\pi^-}m_p}{E_{\pi^-}(\vec{k})E_p(\vec{k})}}\,\Phi_{1s}(\vec{k})\right|^2, \qquad (5.1)$$

where we have neglected the electromagnetic mass differences. The integral over $Q$ is logarithmically divergent. This is very similar to the quantum field theoretic calculation of the Lamb shift [12]. According to Bethe [17] (see also [12]), the integral over $Q$ should be restricted from above by the cut-off $K$. For the calculation of the Lamb shift of hydrogen Bethe set $K$ equal to the mass of the electron, the reduced mass of the $e^- p$ system. Unlike the Lamb shift of the hydrogen the relative momenta of the $\pi^- p$ and $\pi^0 n$ pairs cannot exceed the value $Q\sim1/a_{\mathrm{B}}$. Therefore, for the regularization of the divergent integral we have to set $K = 1/a_{\mathrm{B}} = \alpha\mu$. This yields [2]

$$\int_0^{\infty}\frac{\mathrm{d}Q\,Q^2}{\sqrt{(m_{\pi^-}^2+Q^2)(m_p^2+Q^2)}}$$
$$\times\frac{m_{\pi^-}m_p}{\sqrt{m_{\pi^-}^2+Q^2}+\sqrt{m_p^2+Q^2}-m_{\pi^-}-m_p}\to$$
$$\int_0^{\alpha\mu}\frac{\mathrm{d}Q\,Q^2}{\sqrt{(m_{\pi^-}^2+Q^2)(m_p^2+Q^2)}}$$
$$\times\frac{m_{\pi^-}m_p}{\sqrt{m_{\pi^-}^2+Q^2}+\sqrt{m_p^2+Q^2}-m_{\pi^-}-m_p} = 2\alpha\mu^2. \quad (5.2)$$

---

[2] In the next section we show that the divergent integral can be removed by the renormalization of the reduced mass of the $\pi^- p$ bound system.

Substituting (5.2) in (5.1) we obtain the energy level shift $\epsilon_{1s}^{(2)}$:

$$\epsilon_{1s}^{(2)} = -\alpha \, \frac{4}{3} \left[ 2 \left( a_0^{1/2} \right)^2 + \left( a_0^{3/2} \right)^2 \right]$$
$$\times \left| \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} - m_p}{E_{\pi^-}(\vec{k}) E_p(\vec{k})}} \, \Phi_{1s}(\vec{k}) \right|^2 . \quad (5.3)$$

The second-order contribution to the shift of the energy level of the ground state is negative. This agrees with the theorem of the quantum-mechanical perturbation theory [12].

Neglecting the smearing of the wave function around the origin, which is of order of $O(\alpha)$ (see appendix A), we can rewrite (5.3) as

$$\epsilon_{1s}^{(2)} = -\alpha \, \frac{4}{3} \left[ 2 \left( a_0^{1/2} \right)^2 + \left( a_0^{3/2} \right)^2 \right] |\Psi_{1s}(0)|^2 \quad (5.4)$$

or in the equivalent form

$$-\frac{\epsilon_{1s}^{(2)}}{E_{1s}} = -\frac{8}{3\pi} \frac{1}{a_{\mathrm{B}}^2} \left[ 2 \left( a_0^{1/2} \right)^2 + \left( a_0^{3/2} \right)^2 \right]. \quad (5.5)$$

The width $\Gamma_{1s}$ is defined by (B.11) (see appendix B). Since the contribution of the $\pi^- p$ state to the width $\Gamma_{1s}$ is prohibited kinematically, we obtain

$$\Gamma_{1s} = \frac{1}{m_{\pi^-} m_p} \left[ \frac{8\pi}{3} \left( m_{\pi^-} + m_p \right) \left( a_0^{1/2} - a_0^{3/2} \right) \right]^2$$
$$\times \left| \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} - m_p}{E_{\pi^-}(\vec{k}) E_p(\vec{k})}} \, \Phi_{1s}(\vec{k}) \right|^2$$
$$\times \int \frac{\mathrm{d}^3 Q}{(2\pi)^3 2 E_{\pi^0}(\vec{Q}) 2 E_n(\vec{Q})}$$
$$\times \pi \, \delta(E_{\pi^0}(\vec{Q}) + E_n(\vec{Q}) - m_{\pi^-} - m_p). \quad (5.6)$$

The integral over $\vec{Q}$ is equal to

$$\int \frac{\mathrm{d}^3 Q}{(2\pi)^3 2 E_{\pi^0}(\vec{Q}) 2 E_n(\vec{Q})}$$
$$\times \pi \delta(E_{\pi^0}(\vec{Q}) + E_n(\vec{Q}) - m_{\pi^-} - m_p) = \frac{\mu}{m_{\pi^-} m_p} \frac{p^*}{8\pi}, \quad (5.7)$$

where $p^*$ is defined by (1.3). Substituting (5.7) in (5.6) we obtain the width of the energy level of the ground state of pionic hydrogen

$$\Gamma_{1s} = \frac{8\pi}{9} \left( a_0^{1/2} - a_0^{3/2} \right)^2 \frac{p^*}{\mu}$$
$$\times \left| \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} - m_p}{E_{\pi^-}(\vec{k}) E_p(\vec{k})}} \, \Phi_{1s}(\vec{k}) \right|^2 . \quad (5.8)$$

Using (4.7) we transcribe the r.h.s. of (5.8) into the form

$$\Gamma_{1s} = \frac{8\pi}{9} \left( a_0^{1/2} - a_0^{3/2} \right)^2 \frac{p^*}{\mu} |\Psi_{1s}(0)|^2 \left( 1 + \delta_{1s}^{(\mathrm{s})} \right) \quad (5.9)$$

or in the equivalent form

$$-\frac{\Gamma_{1s}}{E_{1s}} = \frac{16}{9} \frac{p^*}{a_{\mathrm{B}}} \left( a_0^{1/2} - a_0^{3/2} \right)^2 \left( 1 + \delta_{1s}^{(\mathrm{s})} \right) \quad (5.10)$$

with $\delta_{1s}^{(\mathrm{s})}$ given by (4.8).

The partial width $\Gamma_{1s}$, defined by (5.10), is the generalization of the DGBT formula due to the non-perturbative and model-independent contribution caused by the quantum field theoretic and relativistic covariant approach. This correction makes up about 1%. The account for this correction is important for the precision of the extraction of the $S$-wave scattering lengths from the experimental data by the PSI Collaboration. Remind that the precision of the measurement of $\Gamma_{1s}$ is 1% [5].

## 6 Renormalization of the reduced mass of pionic hydrogen and finiteness of $\epsilon_{1s}^{(2)}$

We have found that the second-order contribution to the shift of the energy level $\epsilon_{1s}^{(2)}$ diverges logarithmically. The appearance of divergent contributions to the shifts of the energy levels of hydrogen-like atoms is a well-known phenomenon which spans about 60 years since the pioneering paper by Bethe in 1947 [17] who adopted Kramers's principle [19] of the renormalization of the electron mass to the removal of ultra-violet divergences of the second-order contribution to the shift of the energy level of the $2s$ state of hydrogen (see also [20,21,18]).

The Hamilton operator of pionic hydrogen is given by [18]

$$\hat{H}_{A_{\pi p}} = \frac{\hat{\vec{p}}^2}{2\mu_0} - \frac{\alpha}{r} + H_{\mathrm{str}}, \quad (6.1)$$

where $\hat{\vec{p}} = -i \, \nabla$ is the operator of the relative motion of the $\pi^- p$ system and $r$ is the relative distance, $\mu_0$ is the *bare* reduced mass and $H_{\mathrm{str}} = -\int \mathrm{d}^3 x \, \mathcal{L}_{\mathrm{str}}(x)$. Introducing the physical renormalized reduced mass $\mu_0 = \mu - \delta\mu$, we can rewrite the Hamilton operator (6.1) as follows [18] [3]:

$$\hat{H}_{A_{\pi p}} = \frac{\hat{\vec{p}}^2}{2(\mu - \delta\mu)} - \frac{\alpha}{r} + H_{\mathrm{str}} =$$
$$\frac{\hat{\vec{p}}^2}{2\mu} - \frac{\alpha}{r} + H_{\mathrm{str}} + \delta\mu \, \frac{\hat{\vec{p}}^2}{2\mu^2}. \quad (6.2)$$

The energy of the ground state calculated up to the second order in strong low-energy interaction reads

$$E_{1s} = E_{1s}^{(0)} + \epsilon_{1s}^{(1)} + \epsilon_{1s}^{(2)} + \frac{\delta\mu}{2\mu^2} \left\langle 1s | \hat{\vec{p}}^2 | 1s \right\rangle, \quad (6.3)$$

where $E_{1s}^{(0)} = -\alpha/2a_{\mathrm{B}} = -\alpha^2 \mu/2$. The last term in (6.3) is equal to

$$\frac{\delta\mu}{2\mu^2} \left\langle 1s | \hat{\vec{p}}^2 | 1s \right\rangle = \frac{\delta\mu}{2\mu^2 a_{\mathrm{B}}^2}. \quad (6.4)$$

---

[3] Since we do not calculate closed hadron loops caused by $H_{\mathrm{str}} = -\int \mathrm{d}^3 x \, \mathcal{L}_{\mathrm{str}}(x)$, the parameters of strong interactions are left unrenormalized and equal to measured values.

Following [18] the renormalization of the mass $\delta\mu$ should cancel the divergent part of $\epsilon_{1s}^{(2)}$ (5.1). This yields

$$
\begin{aligned}
\delta\mu = {}& \frac{4}{3}\, a_{\mathrm{B}}^2 \left[ 2\left(a_0^{1/2}\right)^2 + \left(a_0^{3/2}\right)^2 \right] \\
& \times \int_K^\infty \frac{\mathrm{d}Q\, Q^2}{\sqrt{(m_{\pi^-}^2 + Q^2)(m_p^2 + Q^2)}} \\
& \times \frac{m_{\pi^-} m_p}{\sqrt{m_{\pi^-}^2 + Q^2} + \sqrt{m_p^2 + Q^2} - m_{\pi^-} - m_p} \\
& \times \left| \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(\vec{k}) E_p(\vec{k})}}\, \Phi_{1s}(\vec{k}) \right|^2 ,
\end{aligned}
\tag{6.5}
$$

where $K$ is the cut-off. Hence, the renormalized shift of the energy level is given by

$$
\epsilon_{1s} = \epsilon_{1s}^{(1)} + \epsilon_{1s}^{(2)}(K),
\tag{6.6}
$$

where $\epsilon_{1s}^{(1)}$ is defined by (4.6) and $\epsilon_{1s}^{(2)}(K)$ reads

$$
\begin{aligned}
\epsilon_{1s}^{(2)}(K) = {}& -\frac{2}{3}\, \frac{1}{\mu^2} \left[ 2\left(a_0^{1/2}\right)^2 + \left(a_0^{3/2}\right)^2 \right] \\
& \times \int_0^K \frac{\mathrm{d}Q\, Q^2}{\sqrt{(m_{\pi^-}^2 + Q^2)(m_p^2 + Q^2)}} \\
& \times \frac{m_{\pi^-} m_p}{\sqrt{m_{\pi^-}^2 + Q^2} + \sqrt{m_p^2 + Q^2} - m_{\pi^-} - m_p} \\
& \times \left| \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(\vec{k}) E_p(\vec{k})}}\, \Phi_{1s}(\vec{k}) \right|^2 .
\end{aligned}
\tag{6.7}
$$

Since the relative momenta of the $\pi^- p$ and $\pi^0 n$ pairs cannot exceed the value $1/a_{\mathrm{B}}$, we have to set $K = 1/a_{\mathrm{B}} = \alpha\mu$. This gives the expression (5.3).

We would like to emphasize that the integral over $Q$ depends substantially on the ultra-violet cut-off $K$ even if $K \ll m_p$. Indeed, in the limit $m_p \to \infty$ the result of the regularization of the integral over $Q$ reads

$$
\begin{aligned}
& \int_0^\infty \frac{\mathrm{d}Q\, Q^2}{\sqrt{(m_{\pi^-}^2 + Q^2)(m_p^2 + Q^2)}} \\
& \times \frac{m_{\pi^-} m_p}{\sqrt{m_{\pi^-}^2 + Q^2} + \sqrt{m_p^2 + Q^2} - m_{\pi^-} - m_p} \to \\
& \int_0^K \frac{\mathrm{d}Q\, Q^2}{\sqrt{m_{\pi^-}^2 + Q^2}} \frac{m_{\pi^-}}{\sqrt{m_{\pi^-}^2 + Q^2} - m_{\pi^-}} = \\
& m_{\pi^-} K \left[ 1 + \frac{m_{\pi^-}}{K} \ell n \left( \frac{K}{m_{\pi^-}} + \sqrt{1 + \frac{K^2}{m_{\pi^-}^2}} \right) \right].
\end{aligned}
\tag{6.8}
$$

Setting, for example, $K = \mu$, we get $1.904\,\mu^2$ instead of $2\alpha\mu^2$ (5.2). This increases the contribution of strong low-energy interactions to the second order of the perturbation theory by a factor of 131. The former contradicts the

experimental data [5,9] and confirms our choice of the cut-off, $K = \alpha\mu$.

We would like to emphasize that this does not mean that we tune the value of the cut-off to fit the experimental data of the shift of the energy level of pionic hydrogen [9]. This implies only that, in agreement with our assumption, the experimental data on the energy level displacement of the ground state of pionic hydrogen testify the impossibility for the pair $\pi^- p$, bound by Coulombic force, to have virtual momenta greater than $\alpha\mu$.

In sect. 8 we calculate a numerical value of $\epsilon_{1s}^{(2)}$ relative to $\epsilon_{1s}^{(1)}$, $\epsilon_{1s}^{(2)}/\epsilon_{1s}^{(1)} = 1.11 \times 10^{-3}$, which agrees numerically with the result obtained by Trueman, $|\epsilon_{1s}^{(2)}/\epsilon_{1s}^{(1)}| \sim a_{\pi^- p}/a_{\mathrm{B}} = \alpha\mu\left(2a_0^{1/2} + a_0^{3/2}\right) = 1.68 \times 10^{-3}$, within the non-relativistic potential model approach [1]. Such an agreement is in favour of our choice of the cut-off, $K = \alpha\mu$.

# 7 Generalization of the DGBT formulas

Summarizing the results obtained in preceding sections we get the total shift and width of the energy level of the ground state of pionic hydrogen:

$$
\begin{aligned}
\epsilon_{1s} = {}& -\frac{2\pi}{3}\, \frac{1}{\mu} \left\{ \left(2a_0^{1/2} + a_0^{3/2}\right) + \frac{2\alpha}{\pi}\, \mu \left( 2\left(a_0^{1/2}\right)^2 + \left(a_0^{3/2}\right)^2 \right) \right\} \\
& \times \left| \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(\vec{k}) E_p(\vec{k})}}\, \Phi_{1s}(\vec{k}) \right|^2 , \\
\Gamma_{1s} = {}& \frac{8\pi}{9} \left( a_0^{1/2} - a_0^{3/2} \right)^2 \frac{p^*}{\mu} \\
& \times \left| \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(\vec{k}) E_p(\vec{k})}}\, \Phi_{1s}(\vec{k}) \right|^2 .
\end{aligned}
\tag{7.1}
$$

These are the DGBT formulas generalized by i) the non-perturbative corrections caused by a quantum field theoretic and relativistic covariant approach, leading to the smearing of the wave function of the ground state of pionic hydrogen around the origin $r = 0$, and ii) the second-order correction of the perturbation theory in strong low-energy interactions.

The formulas (7.1) can be rewritten as

$$
\begin{aligned}
\epsilon_{1s} &= -\frac{2\pi}{3}\, \frac{1}{\mu} \left( 2a_0^{1/2} + a_0^{3/2} \right) |\Psi_{1s}(0)|^2 \left( 1 + \delta_{1s}^{(s)} + \delta_{1s}^{(2)} \right), \\
\Gamma_{1s} &= \frac{8\pi}{9} \left( a_0^{1/2} - a_0^{3/2} \right)^2 \frac{p^*}{\mu} |\Psi_{1s}(0)|^2 \left( 1 + \delta_{1s}^{(s)} \right)
\end{aligned}
\tag{7.2}
$$

or in the equivalent form

$$
\begin{aligned}
-\frac{\epsilon_{1s}}{E_{1s}} &= -\frac{4}{3}\, \frac{1}{a_{\mathrm{B}}} \left( 2a_0^{1/2} + a_0^{3/2} \right) \left( 1 + \delta_{1s}^{(s)} + \delta_{1s}^{(2)} \right), \\
-\frac{\Gamma_{1s}}{E_{1s}} &= \frac{16}{9}\, \frac{p^*}{a_{\mathrm{B}}} \left( a_0^{1/2} - a_0^{3/2} \right)^2 \left( 1 + \delta_{1s}^{(s)} \right),
\end{aligned}
\tag{7.3}
$$

where $\delta_{1s}^{(s)}$ is given by (4.8) and $\delta_{1s}^{(2)}$ is defined by

$$\delta_{1s}^{(2)} = \frac{2}{\pi} \frac{1}{a_{\mathrm{B}}} \frac{2\left(a_0^{1/2}\right)^2 + \left(a_0^{3/2}\right)^2}{2a_0^{1/2} + a_0^{3/2}} =$$
$$\frac{2\alpha}{\pi} \mu \frac{2\left(a_0^{1/2}\right)^2 + \left(a_0^{3/2}\right)^2}{2a_0^{1/2} + a_0^{3/2}}. \qquad (7.4)$$

Formulas (7.3) for the displacement of the energy level of the ground state of pionic hydrogen should be applied to a theoretical analysis of experimental data by the PSI Collaboration [5].

## 8 Theoretical accuracy of the DGBT formulas

The displacement of the energy level of the ground state of pionic hydrogen, caused by strong low-energy interactions, has been recently measured by the PSI Collaboration [9].The results read

$$\epsilon_{1s} = -7.108 \pm 0.013\,(\text{stat.}) \pm 0.034\,(\text{syst.})\,\text{eV},$$
$$\Gamma_{1s} = 0.868 \pm 0.040\,(\text{stat.}) \pm 0.038\,(\text{syst.})\,\text{eV}. \qquad (8.1)$$

This gives the experimental values of the $S$-wave scattering lengths [9]

$$a_{\pi^- p \to \pi^- p} = +0.0883 \pm 0.0008\,m_{\pi^-}^{-1},$$
$$a_{\pi^- p \to \pi^0 n} = -0.1280 \pm 0.0060\,m_{\pi^-}^{-1}, \qquad (8.2)$$

which were obtained by fitting the DGBT formulas (1.5)[4]. For the $S$-wave scattering lengths $a_0^{1/2}$ and $a_0^{3/2}$ we obtain

$$a_0^{1/2} = +0.1788 \pm 0.0043\,m_{\pi^-}^{-1},$$
$$a_0^{3/2} = -0.0927 \pm 0.0085\,m_{\pi^-}^{-1}. \qquad (8.3)$$

We would like to emphasize that the experimental values of the $S$-wave scattering lengths (8.3) satisfy Adler's consistency condition [10,11]

$$a_0^{1/2} + 2a_0^{3/2} = 0. \qquad (8.4)$$

The experimental value is $a_0^{1/2} + 2a_0^{3/2} = -0.0066 \pm 0.0175\,m_{\pi^-}^{-1}$.

Let us now estimate the contribution of the second-order correction $\epsilon_{1s}^{(2)}$ relative to $\epsilon_{1s}^{(1)}$. Using (5.3) and (4.6) we get

$$\frac{\epsilon_{1s}^{(2)}}{\epsilon_{1s}^{(1)}} = \frac{2\alpha}{\pi} \mu \frac{2\left(a_0^{1/2}\right)^2 + \left(a_0^{3/2}\right)^2}{2a_0^{1/2} + a_0^{3/2}} =$$
$$(1.11 \pm 0.06) \times 10^{-3} = (0.111 \pm 0.006)\%. \qquad (8.5)$$

---

[4] The electromagnetic corrections [8] have been also taken into account [9].

This testifies that strong low-energy interactions can be treated perturbatively for the analysis of the energy level displacement of the ground state of pionic hydrogen.

The theoretical accuracy of the DGBT formula for the shift of the energy level $\epsilon_{1s}$, relative to the expression given by (7.3), is defined by

$$\delta_{1s} = \delta_{1s}^{(s)} + \delta_{1s}^{(2)} = -9.69 \times 10^{-3} + (1.11 \pm 0.06) \times 10^{-3} =$$
$$(-8.58 \pm 0.06) \times 10^{-3}. \qquad (8.6)$$

The first term, $\delta_{1s}^{(s)} = -9.69 \times 10^{-3}$, does not depend on the $S$-wave scattering lengths. This is the non-perturbative and model-independent correction caused by the relativistic factor $\sqrt{m_{\pi^-} m_p / E_{\pi^-}(k) E_p(k)}$, smearing the wave function of pionic hydrogen around the origin $r = 0$. The second term, $\delta_{1s}^{(2)} = (1.11 \pm 0.06) \times 10^{-3}$, depends explicitly on the $S$-wave scattering lengths. It is defined by strong low-energy interactions to the second order of the perturbation theory. We would like to emphasize that the correction $\delta_{1s}^{(2)}$ is model-independent as well as $\delta_{1s}^{(s)}$.

The width of the energy level $\Gamma_{1s}$, given by (7.3), is valid to any order of the perturbation theory in strong low-energy interactions. Indeed, the contribution of the next-to-leading corrections should be related to the $\pi^0 n$ rescattering in the final state [22]. The width $\Gamma_{1s}$, modified by the inclusion of the $\pi^0 n$ rescattering, reads [22]

$$-\frac{\Gamma_{1s}}{E_{1s}} = \frac{16}{9} \frac{\left(a_0^{1/2} - a_0^{3/2}\right)^2}{1 + \frac{1}{9} p^{*2} \left(a_0^{1/2} + 2a_0^{3/2}\right)^2} \frac{p^*}{a_{\mathrm{B}}} \left(1 + \delta_{1s}^{(s)}\right). \qquad (8.7)$$

Due to Adler's consistency condition (8.4) the contribution of the $\pi^0 n$ rescattering vanishes and we arrive at the expression (7.3). Of course, this assertion does not concern electromagnetic corrections which should be taken into account as has been done for the pionium, the bound $\pi^+ \pi^-$ state, by Gasser *et al.* [23].

Hence, one can argue that the width of the energy level of the ground state of pionic hydrogen given by (7.3) is valid to all orders of the perturbation theory in strong low-energy interactions. Therefore, the theoretical accuracy of the DGBT formula for the width (1.5) is defined by the non-perturbative correction caused by the smearing of the wave function around the origin, *i.e.* $\delta_{1s}^{(s)} = -9.69 \times 10^{-3}$.

## 9 Conclusion

The revision of the DGBT formulas, derived in the middle of the last century within a non-relativistic potential approach, is motivated by the contemporary level of the development of experimental and theoretical physics. The possibility to measure the displacement of the energy level of the ground state of pionic hydrogen with the accuracy 0.2% for the shift and 1% for the width, reached by the PSI Collaboration [5], imposes new strict requirements on theoretical formulas.

The derivation of the energy level displacement of the ground state of pionic hydrogen within a quantum field theoretic and relativistic covariant approach, developed above, has led to the corrected DGBT formulas

$$-\frac{\epsilon_{1s}}{E_{1s}} = -\frac{4}{3}\frac{1}{a_{\mathrm{B}}}\left(2a_0^{1/2}+a_0^{3/2}\right)\left(1+\delta_{1s}^{(\mathrm{s})}+\delta_{1s}^{(2)}\right) =$$
$$-\frac{4}{3}\frac{1}{a_{\mathrm{B}}}\left(2a_0^{1/2}+a_0^{3/2}\right)\left(1+(-8.58\pm0.06)\times10^{-3}\right),$$
$$-\frac{\Gamma_{1s}}{E_{1s}} = \frac{16}{9}\frac{p^*}{a_{\mathrm{B}}}\left(a_0^{1/2}-a_0^{3/2}\right)^2\left(1+\delta_{1s}^{(\mathrm{s})}\right) =$$
$$\frac{16}{9}\frac{p^*}{a_{\mathrm{B}}}\left(a_0^{1/2}-a_0^{3/2}\right)^2\left(1+(-9.69)\times10^{-3}\right). \quad (9.1)$$

Numerically the deviation from the DGBT formulas makes up about 1%. Nevertheless, this is very important for the precision of the extraction of the $S$-wave scattering lengths from the experimental data on $\epsilon_{1s}$ and $\Gamma_{1s}$.

Recall that the experimental accuracy of $\epsilon_{1s}$ and $\Gamma_{1s}$, measured by the PSI Collaboration [9] (8.1), is 0.5% and 4.6%, respectively. The experimental accuracy, which can be reached by the PSI Collaboration [5] in the new set of experiments, is expected to be equal to 0.2% and 1% for $\epsilon_{1s}$ and $\Gamma_{1s}$, respectively.

The obtained deviation from the DGBT formulas is defined by i) the non-perturbative correction $\delta_{1s}^{(\mathrm{s})} = -9.69 \times 10^{-3}$, caused by the smearing of the wave function $\Psi_{1s}(0)$ of pionic hydrogen around the origin $r = 0$ due to the relativistic factor

$$\Psi_{1s}(0) \rightarrow \int \frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{m_\pi-m_p}{E_{\pi^-}(\vec{k})E_p(\vec{k})}}\,\Phi_{1s}(\vec{k}) =$$
$$\Psi_{1s}(0)\left(1+\frac{1}{2}\,\delta_{1s}^{(\mathrm{s})}\right), \quad (9.2)$$

and ii) the perturbative correction $\delta_{1s}^{(2)} = (1.11 \pm 0.06) \times 10^{-3}$ calculated to the second order in strong low-energy interactions. We would like to emphasize that both $\delta_{1s}^{(\mathrm{s})}$ and $\delta_{1s}^{(2)}$ are model-independent corrections.

The value of $\delta_{1s}^{(2)}$ is a test of the applicability of the perturbative treatment of strong low-energy interactions to the analysis of the displacements of the energy levels of pionic hydrogen. It is important to notice that, due to Adler's consistency condition, the partial width of the energy level of the ground state (9.1) is valid to any order of the perturbative theory in strong low-energy interactions.

We have found that the second-order correction $\epsilon_{1s}^{(2)}$ to the shift of the energy level is of order $O\left(2\left(a_0^{1/2}\right)^2+\left(a_0^{3/2}\right)^2\right)$ and divergent logarithmically. Following the experience of the theoretical analysis of the Lamb shift by Bethe and removing logarithmic divergence by renormalization of the reduced mass of the bound $\pi^-p$ system we obtain the second-order contribution to the energy shift dependent on the cut-off $K$: $\epsilon_{1s}^{(2)}(K)$. Since by derivation relative momenta of the $\pi^-p$ and $\pi^0n$ pairs cannot exceed the value $1/a_{\mathrm{B}} = \alpha\mu$, we have set $K = 1/a_{\mathrm{B}} = \alpha\mu$.

We would like to emphasize that the effective Lagrangian $\mathcal{L}_{\mathrm{str}}(0)$, which we have used for the description of the amplitude of the $\pi N$ scattering, is unrenormalizable and depends only on physical parameters. Perturbative corrections to the energy level displacements can be calculated only in the tree-approximation and no hadronic loops are allowed. This means that all divergent contributions, which we encounter for the calculation of the shift of the energy level, should be removed by renormalization of the reduced mass of the bound $\pi^-p$ state.

Setting $K = \alpha\mu$ we have obtained the value of the second-order correction $\epsilon_{1s}^{(2)}(\alpha\mu)$, which makes up $(0.111\pm 0.006)\%$ of the first-order correction $\epsilon_{1s}^{(1)}$. This testifies the applicability of the perturbative treatment of strong low-energy interactions to the analysis of the displacements of the energy levels of pionic hydrogen. We have shown that the width $\Gamma_{1s}$ of the energy level of the ground state, defined by the DGBT formula with the wave function smeared around the origin (9.2) due to the relativistic factor, is valid to any order of the perturbation theory due to Adler's consistency condition.

In our approach the relative correction $\epsilon_{1s}^{(2)}/\epsilon_{1s}^{(1)} = 1.11\times10^{-3}$ is fully due to the choice of the cut-off, $K = \alpha\mu$. Any higher values of $K$ can change this ratio drastically. Since, as usual, the choice of the cut-off is the most subtle problem of quantum field theory, one has to look for arguments in favour of the given choice. In this connection we would like to refer to the result obtained by Trueman within the non-relativistic potential model approach [1]. Trueman calculated the second-order correction to the shift of the energy level. In our notation his result reads $\left|\epsilon_{1s}^{(2)}/\epsilon_{1s}^{(1)}\right| \simeq a_{\pi^-p}/a_{\mathrm{B}} = \alpha\mu\left(2a_0^{1/2}+a_0^{3/2}\right) = 1.68 \times 10^{-3}$. Such a numerical agreement is in favour of our choice of the cut-off, $K = \alpha\mu$.

Our quantum field theoretic and relativistic covariant treatment of the shift and width of the ground state of pionic hydrogen allows to calculate the shift $\epsilon_{n\ell}$ and width $\Gamma_{n\ell}$ of the energy level of any excited state $n\ell$ of pionic hydrogen with $n \neq 1$ and $\ell \neq 0$. In our approach for $\epsilon_{n\ell}$ and $\Gamma_{n\ell}$ with $n \neq 1$ and $\ell \neq 0$ we expect to get the following results:

$$\epsilon_{n\ell} \sim a_{\pi^-p\rightarrow\pi^-p}^{(\ell)}\left|\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{m_\pi-m_p}{E_{\pi^-}(k)E_p(k)}}\,k^\ell\,\Phi_{n\ell}(k)\right|^2,$$
$$\Gamma_{n\ell} \sim \left|a_{\pi^-p\rightarrow\pi^0n}^{(\ell)}\right|^2\left|\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{m_\pi-m_p}{E_{\pi^-}(k)E_p(k)}}\,k^\ell\,\Phi_{n\ell}(k)\right|^2,$$
$$(9.3)$$

where $a_{\pi^-p\rightarrow\pi^-p}^{(\ell)}$ and $a_{\pi^-p\rightarrow\pi^0n}^{(\ell)}$ are the scattering lengths of the $\pi N$ scattering with a relative angular momentum $\ell$, $\Phi_{n\ell}(k)$ is the radial wave function of the $n\ell$ state of pionic hydrogen in the momentum representation.

A more detailed calculation of $\epsilon_{n\ell}$ and $\Gamma_{n\ell}$ with $n \neq 1$ and $\ell \neq 0$ including corrections caused by the QCD isospin-breaking and electromagnetic interactions we are planning to carry out in our forthcoming publication.

The results, which we expect to obtain, should be compared with those by Lyubovitskij and Rusetsky [6] and Lyubovitskij *et al.* [7] derived for the ground state of pionic hydrogen.

According to Lyubovitskij and Rusetsky [6] and Lyubovitskij *et al.* [7], the DGBT formula for the shift of the energy level of the ground state is modified by the contributions of the QCD isospin-breaking and electromagnetic interactions by the factor $(1 + \tilde{\delta}_{1s})$ [6,7]. The estimates of the correction $\tilde{\delta}_{1s}$ obtained within ChPT [6] and the perturbative chiral quark model (PChQM) [7] are equal to $\tilde{\delta}_{1s}^{(\mathrm{ChPT})} = (-4.8 \pm 2.0) \times 10^{-2}$ and $\tilde{\delta}_{1s}^{(\mathrm{PChQM})} = -2.8 \times 10^{-2}$, respectively. These results are in agreement with that calculated within the potential model approach (PMA) [8]: $\tilde{\delta}_{1s}^{(\mathrm{PMA})} = (-2.1 \pm 0.5) \times 10^{-2}$.

We have shown that the corrections to the shift of the energy level of the ground state of pionic hydrogen caused by strong low-energy interactions are of order of magnitude smaller compared with the corrections induced by the QCD isospin-breaking and electromagnetic interactions [6,7]. Nevertheless, the non-perturbative correction, defined by the smearing of the wave function of the ground state of pionic hydrogen around the origin (9.2), is co-measurable with the contributions caused by QCD isospin-breaking and electromagnetic interactions [6] and [7]. Such a correction calculated for the energy level displacement of the ground state of pionium, the $\pi^-\pi^+$ bound state, is equal to $\delta_{1s}^{(\mathrm{s})} = -4\alpha/\pi = -9.29 \times 10^{-3}$.

# Appendix A. Calculation of the momentum integral in (4.6)

The momentum integral (4.6) defines the generalization of the DGBT formulas due to the quantum field theoretic and relativistic covariant derivation. For comparison of the obtained result with the DGBT formulas the momentum integral (4.6) should be calculated explicitly. The calculation of this momentum integral we carry out in the limit $m_p \to \infty$. This yields

$$
\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} \, \Phi_{1s}(\vec{k}) =
$$
$$
\int \mathrm{d}^3 x \, \Psi_{1s}(\vec{r}) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} \, e^{i \vec{k} \cdot \vec{r}} =
$$
$$
-\Psi_{1s}(0) \sqrt{m_{\pi^-}} \frac{2}{\pi} \int_0^\infty \mathrm{d}r \, r \, e^{-r/a_{\mathrm{B}}} \frac{\mathrm{d}}{\mathrm{d}r} \int_0^\infty \frac{\mathrm{d}k \cos(kr)}{(m_{\pi^-}^2 + k^2)^{1/4}}. \tag{A.1}
$$

The integral over $k$ is equal to [24]

$$
\int_0^\infty \frac{\mathrm{d}k \cos(kr)}{(m_{\pi^-}^2 + k^2)^{1/4}} = \frac{2^{1/4} \sqrt{\pi}}{\Gamma(1/4)} \frac{\sqrt{m_{\pi^-}}}{(m_{\pi^-} r)^{1/4}} \, K_{1/4}(m_{\pi^-} r), \tag{A.2}
$$

where we have used the formula [25]

$$
K_\nu(xz) = \Gamma\left(\nu + \frac{1}{2}\right) \frac{(2z)^\nu}{\sqrt{\pi} \, x^\nu} \int_0^\infty \frac{\cos(xt)\mathrm{d}t}{(t^2 + z^2)^{\nu+1/2}} \tag{A.3}
$$

and the relation $K_{-\nu}(xz) = K_\nu(xz)$ [26].

Substituting (A.2) in (A.1) we get

$$
\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} \, \Phi_{1s}(\vec{k}) = -\Psi_{1s}(0) \frac{1}{\sqrt{\pi}} \frac{2^{5/4}}{\Gamma(1/4)}
$$
$$
\times \int_0^\infty \mathrm{d}x \, x \, e^{-x/m_{\pi^-} a_{\mathrm{B}}} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1/4} K_{1/4}(x)\right), \tag{A.4}
$$

where $x = m_{\pi^-} r$. Using the formula [27]

$$
\left(\frac{\mathrm{d}}{\mathrm{d}x} x^{-1/4} K_{1/4}(x)\right) = -x^{-1/4} K_{5/4}(x) \tag{A.5}
$$

we transform the integral over $x$ to the form

$$
\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} \, \Phi_{1s}(\vec{k}) = \Psi_{1s}(0) \frac{1}{\sqrt{\pi}} \frac{2^{5/4}}{\Gamma(1/4)}
$$
$$
\times \int_0^\infty \mathrm{d}x \, e^{-x/m_{\pi^-} a_{\mathrm{B}}} x^{3/4} K_{1/4}(x). \tag{A.6}
$$

Using the formula [28]

$$
\int_0^\infty \mathrm{d}x \, x^\mu \, K_\nu(x) = 2^{\mu-1} \, \Gamma\left(\frac{\mu+\nu+1}{2}\right) \Gamma\left(\frac{\mu-\nu+1}{2}\right) \tag{A.7}
$$

we get

$$
\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} \, \Phi_{1s}(\vec{k}) =
$$
$$
\Psi_{1s}(0)\left(1 - \alpha \frac{\mu}{m_{\pi^-}} \frac{4}{\sqrt{\pi}} \frac{\Gamma(3/4)}{\Gamma(1/4)} + O(\alpha^2)\right) =
$$
$$
\Psi_{1s}(0)\left(1 + \frac{1}{2} \delta_{1s}^{(\mathrm{s})}\right). \tag{A.8}
$$

The correction $\delta_{1s}^{(\mathrm{s})}$, caused by the smearing of the wave function of the ground state of pionic hydrogen by the relativistic factor, reads

$$
\delta_{1s}^{(\mathrm{s})} = \frac{1}{|\Psi_{1s}(0)|^2}
$$
$$
\times \left(\left|\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} \, \Phi_{1s}(\vec{k})\right|^2 - |\Psi_{1s}(0)|^2\right) =
$$
$$
-\alpha \frac{\mu}{m_{\pi^-}} \frac{8}{\sqrt{\pi}} \frac{\Gamma(3/4)}{\Gamma(1/4)} + O(\alpha^2) = -9.69 \times 10^{-3}. \tag{A.9}
$$

Thus, the correction to the DGBT formulas, caused the quantum field theoretic and relativistic covariant derivation of the energy level displacement of the ground state of pionic hydrogen, makes up 0.969%.

# Appendix B. Calculation of $\epsilon_{1s}^{(2)}$ and $\Gamma_{1s}$

The energy level shift $\epsilon_{1s}^{(2)}$ and the width $\Gamma_{1s}$ are defined by the second term in (3.8). We rewrite this term as follows:

$$
\frac{i}{2} \int \mathrm{d}^4 x \, \langle A_{\pi p}^{(1s)}(\vec{P}, \sigma_p) | \mathrm{T}(\mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0)) | A_{\pi p}^{(1s)}(\vec{P}, \sigma_p) \rangle =
$$
$$
\frac{i}{2} \int \mathrm{d}^4 x \, \theta(x^0)
$$
$$
\times \langle A_{\pi p}^{(1s)}(\vec{P}, \sigma_p) | \mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0) | A_{\pi p}^{(1s)}(\vec{P}, \sigma_p) \rangle
$$
$$
+ \frac{i}{2} \int \mathrm{d}^4 x \, \theta(-x^0)
$$
$$
\times \langle A_{\pi p}^{(1s)}(\vec{P}, \sigma_p) | \mathcal{L}_{\mathrm{str}}(0)\mathcal{L}_{\mathrm{str}}(x) | A_{\pi p}^{(1s)}(\vec{P}, \sigma_p) \rangle =
$$
$$
\frac{i}{2} \int \mathrm{d}^4 x \, \theta(x^0) \frac{1}{(2\pi)^6} \int \frac{\mathrm{d}^3 k_{\pi^-}}{\sqrt{2E_{\pi^-}(\vec{k}_{\pi^-})}} \frac{\mathrm{d}^3 k_p}{\sqrt{2E_p(\vec{k}_p)}}
$$
$$
\times \frac{\mathrm{d}^3 q_{\pi^-}}{\sqrt{2E_{\pi^-}(\vec{q}_{\pi^-})}} \frac{\mathrm{d}^3 q_p}{\sqrt{2E_p(\vec{q}_p)}} \, \delta^{(3)}(\vec{P} - \vec{k}_{\pi^-} - \vec{k}_p)
$$
$$
\times \delta^{(3)}(\vec{P} - \vec{q}_{\pi^-} - \vec{q}_p) \, 2E_A^{(1s)}(\vec{P}) \, \Phi_{1s}^\dagger(\vec{k}_{\pi^-}) \, \Phi_{1s}(\vec{q}_{\pi^-})
$$
$$
\times \langle \pi^-(\vec{k}_{\pi^-})p(\vec{k}_p, \sigma_p) | \mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0) | \pi^-(\vec{q}_{\pi^-})p(\vec{q}_p, \sigma_p) \rangle
$$
$$
\times U_{1s}(\vec{P}, \sigma_p) + \frac{i}{2} \int \mathrm{d}^4 x \, \theta(-x^0) \frac{1}{(2\pi)^6} \int \frac{\mathrm{d}^3 k_{\pi^-}}{\sqrt{2E_{\pi^-}(\vec{k}_{\pi^-})}}
$$
$$
\times \frac{\mathrm{d}^3 k_p}{\sqrt{2E_p(\vec{k}_p)}} \frac{\mathrm{d}^3 q_{\pi^-}}{\sqrt{2E_{\pi^-}(\vec{q}_{\pi^-})}} \frac{\mathrm{d}^3 q_p}{\sqrt{2E_p(\vec{q}_p)}} \, \delta^{(3)}(\vec{P} - \vec{k}_{\pi^-} - \vec{k}_p)
$$
$$
\times \delta^{(3)}(\vec{P} - \vec{q}_{\pi^-} - \vec{q}_p) \, 2E_A^{(1s)}(\vec{P}) \, \Phi_{1s}^\dagger(\vec{k}_{\pi^-}) \, \Phi_{1s}(\vec{q}_{\pi^-})
$$
$$
\times \langle \pi^-(\vec{k}_{\pi^-})p(\vec{k}_p, \sigma_p) | \mathcal{L}_{\mathrm{str}}(0)\mathcal{L}_{\mathrm{str}}(x) | \pi^-(\vec{q}_{\pi^-})p(\vec{q}_p, \sigma_p) \rangle.
$$
$$
\text{(B.1)}
$$

Setting $\vec{P} = 0$ and making the necessary integration we get

$$
\frac{i}{2} \int \mathrm{d}^4 x \, \langle A_{\pi p}^{(1s)}(0, \sigma_p) | \mathrm{T}(\mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0)) | A_{\pi p}^{(1s)}(0, \sigma_p) \rangle =
$$
$$
2M_A^{(1s)} \frac{i}{2} \int \mathrm{d}^4 x \, \theta(x^0) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\Phi_{1s}^\dagger(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}}
$$
$$
\times \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\Phi_{1s}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}}
$$
$$
\times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0) | \pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rangle
$$
$$
+ 2M_A^{(1s)} \frac{i}{2} \int \mathrm{d}^4 x \, \theta(-x^0) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\Phi_{1s}^\dagger(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}}
$$
$$
\times \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\Phi_{1s}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}}
$$
$$
\times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathcal{L}_{\mathrm{str}}(0)\mathcal{L}_{\mathrm{str}}(x) | \pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rangle.
$$
$$
\text{(B.2)}
$$

According to (3.9) the shift $\epsilon_{1s}^{(2)}$ and the width $\Gamma_{1s}$ are defined by

$$
-\epsilon_{1s}^{(2)} + i \frac{\Gamma_{1s}}{2} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\Phi_{1s}^\dagger(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}}
$$
$$
\times \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\Phi_{1s}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}} \frac{i}{2} \int \mathrm{d}^4 x
$$
$$
\times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathrm{T}(\mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0)) | \pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rangle.
$$
$$
\text{(B.3)}
$$

Since the wave functions $\Phi_{1s}^\dagger(\vec{k})$ and $\Phi_{1s}(\vec{q})$ restrict the momenta $k \sim q \sim 1/a_{\mathrm{B}}$, the matrix elements of the operators $\mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0)$ and $\mathcal{L}_{\mathrm{str}}(0)\mathcal{L}_{\mathrm{str}}(x)$ can be calculated in the low-energy limit $k, q \to 0$.

In the low-energy limit the main contribution to the intermediate states of the matrix elements of the operators $\mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0)$ and $\mathcal{L}_{\mathrm{str}}(0)\mathcal{L}_{\mathrm{str}}(x)$ comes from the states $|\pi^- p\rangle$ and $|\pi^0 n\rangle$. Inserting these intermediate states we get

$$
\frac{i}{2} \int \mathrm{d}^4 x
$$
$$
\times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathrm{T}(\mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0)) | \pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rangle =
$$
$$
\frac{i}{2} \int \mathrm{d}^4 x \, \theta(x^0)
$$
$$
\times \sum_{\lambda_p = \pm 1/2} \int \frac{\mathrm{d}^3 Q_{\pi^-}}{(2\pi)^3 2E_{\pi^-}(\vec{Q}_{\pi^-})} \frac{\mathrm{d}^3 Q_p}{(2\pi)^3 2E_p(\vec{Q}_p)}
$$
$$
\times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathcal{L}_{\mathrm{str}}(x) | \pi^-(\vec{Q}_{\pi^-})p(\vec{Q}_p, \lambda_p) \rangle
$$
$$
\times \langle p(\vec{Q}_p, \lambda_p)\pi^-(\vec{Q}_{\pi^-}) | \mathcal{L}_{\mathrm{str}}(0) | \pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rangle
$$
$$
+ \frac{i}{2} \int \mathrm{d}^4 x \, \theta(-x^0)
$$
$$
\times \sum_{\lambda_p = \pm 1/2} \int \frac{\mathrm{d}^3 Q_{\pi^-}}{(2\pi)^3 2E_{\pi^-}(\vec{Q}_{\pi^-})} \frac{\mathrm{d}^3 Q_p}{(2\pi)^3 2E_p(\vec{Q}_p)}
$$
$$
\times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathcal{L}_{\mathrm{str}}(0) | \pi^-(\vec{Q}_{\pi^-})p(\vec{Q}_p, \lambda_p) \rangle
$$
$$
\times \langle p(\vec{Q}_p, \lambda_p)\pi^-(\vec{Q}_{\pi^-}) | \mathcal{L}_{\mathrm{str}}(x) | \pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rangle
$$
$$
+ \frac{i}{2} \int \mathrm{d}^4 x \, \theta(x^0)
$$
$$
\times \sum_{\lambda_n = \pm 1/2} \int \frac{\mathrm{d}^3 Q_{\pi^0}}{(2\pi)^3 2E_{\pi^0}(\vec{Q}_{\pi^0})} \frac{\mathrm{d}^3 Q_n}{(2\pi)^3 2E_n(\vec{Q}_n)}
$$
$$
\times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathcal{L}_{\mathrm{str}}(x) | \pi^0(\vec{Q}_{\pi^0})n(\vec{Q}_n, \lambda_n) \rangle
$$
$$
\times \langle n(\vec{Q}_n, \lambda_n)\pi^0(\vec{Q}_{\pi^0}) | \mathcal{L}_{\mathrm{str}}(0) | \pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rangle
$$
$$
+ \frac{i}{2} \int \mathrm{d}^4 x \, \theta(-x^0)
$$
$$
\times \sum_{\lambda_n = \pm 1/2} \int \frac{\mathrm{d}^3 Q_{\pi^0}}{(2\pi)^3 2E_{\pi^0}(\vec{Q}_{\pi^0})} \frac{\mathrm{d}^3 Q_n}{(2\pi)^3 2E_n(\vec{Q}_n)}
$$
$$
\times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathcal{L}_{\mathrm{str}}(0) | \pi^0(\vec{Q}_{\pi^0})n(\vec{Q}_n, \lambda_n) \rangle
$$
$$
\times \langle n(\vec{Q}_n, \lambda_n)\pi^0(\vec{Q}_{\pi^0}) | \mathcal{L}_{\mathrm{str}}(x) | \pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rangle. \quad \text{(B.4)}
$$

Making the integration over $x$ we obtain

$$\frac{i}{2}\int \mathrm{d}^4 x$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma_p)|\mathrm{T}(\mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0))|\pi^-(\vec{q})p(-\vec{q},\sigma_p)\rangle =$$
$$\frac{1}{2}\sum_{\lambda_p=\pm 1/2}\int \frac{\mathrm{d}^3 Q_{\pi^-}}{(2\pi)^3 2E_{\pi^-}(\vec{Q}_{\pi^-})}$$
$$\times \frac{\mathrm{d}^3 Q_p}{(2\pi)^3 2E_p(\vec{Q}_p)}(2\pi)^3\delta^{(3)}(\vec{Q}_{\pi^-}+\vec{Q}_p)$$
$$\times \frac{1}{E_{\pi^-}(\vec{Q}_{\pi^-})+E_p(\vec{Q}_p)-E_{\pi^-}(\vec{k})-E_p(\vec{k})-i\varepsilon}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma_p)|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{Q}_{\pi^-})p(\vec{Q}_p,\lambda_p)\rangle$$
$$\times \langle p(\vec{Q}_p,\lambda_p)\pi^-(\vec{Q}_{\pi^-})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma_p)\rangle$$
$$+\frac{1}{2}\sum_{\lambda_p=\pm 1/2}\int \frac{\mathrm{d}^3 Q_{\pi^-}}{(2\pi)^3 2E_{\pi^-}(\vec{Q}_{\pi^-})}\frac{\mathrm{d}^3 Q_p}{(2\pi)^3 2E_p(\vec{Q}_p)}$$
$$\times (2\pi)^3\delta^{(3)}(\vec{Q}_{\pi^-}+\vec{Q}_p)$$
$$\times \frac{1}{E_{\pi^-}(\vec{Q}_{\pi^-})+E_p(\vec{Q}_p)-E_{\pi^-}(\vec{q})-E_p(\vec{q})-i\varepsilon}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{Q}_{\pi^-})p(\vec{Q}_p,\lambda_p)\rangle$$
$$\times \langle p(\vec{Q}_p,\lambda_p)\pi^-(\vec{Q}_{\pi^-})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle$$
$$+\frac{1}{2}\sum_{\lambda_n=\pm 1/2}\int \frac{\mathrm{d}^3 Q_{\pi^0}}{(2\pi)^3 2E_{\pi^0}(\vec{Q}_{\pi^0})}\frac{\mathrm{d}^3 Q_n}{(2\pi)^3 2E_n(\vec{Q}_n)}$$
$$\times (2\pi)^3\delta^{(3)}(\vec{Q}_{\pi^0}+\vec{Q}_n)$$
$$\times \frac{1}{E_{\pi^0}(\vec{Q}_{\pi^0})+E_n(\vec{Q}_n)-E_{\pi^-}(\vec{k})-E_p(\vec{k})-i\varepsilon}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^0(\vec{Q}_{\pi^0})n(\vec{Q}_n,\lambda_n)\rangle$$
$$\times \langle n(\vec{Q}_n,\lambda_n)\pi^0(\vec{Q}_{\pi^0})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle$$
$$+\frac{1}{2}\sum_{\lambda_n=\pm 1/2}\int \frac{\mathrm{d}^3 Q_{\pi^0}}{(2\pi)^3 2E_{\pi^0}(\vec{Q}_{\pi^0})}\frac{\mathrm{d}^3 Q_n}{(2\pi)^3 2E_n(\vec{Q}_n)}$$
$$\times (2\pi)^3\delta^{(3)}(\vec{Q}_{\pi^0}+\vec{Q}_n)$$
$$\times \frac{1}{E_{\pi^0}(\vec{Q}_{\pi^0})+E_n(\vec{Q}_n)-E_{\pi^-}(\vec{q})-E_p(\vec{q})-i\varepsilon}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^0(\vec{Q}_{\pi^0})n(\vec{Q}_n,\lambda_n)\rangle$$
$$\times \langle n(\vec{Q}_n,\lambda_n)\pi^0(\vec{Q}_{\pi^0})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle. \quad (B.5)$$

Integrating over $\vec{Q}_p$ and $\vec{Q}_n$ we reduce the r.h.s. of (B.5) to the expression

$$\frac{i}{2}\int \mathrm{d}^4 x$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathrm{T}(\mathcal{L}_{\mathrm{str}}(x)\mathcal{L}_{\mathrm{str}}(0))|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle =$$
$$\frac{1}{2}\sum_{\lambda_p=\pm 1/2}\int \frac{\mathrm{d}^3 Q}{(2\pi)^3 2E_{\pi^-}(\vec{Q})2E_p(\vec{Q})}$$
$$\times \frac{1}{E_{\pi^-}(\vec{Q})+E_p(\vec{Q})-E_{\pi^-}(\vec{k})-E_p(\vec{k})-i\varepsilon}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{Q})p(-\vec{Q},\lambda_p)\rangle$$

$$\times \langle p(-\vec{Q},\lambda_p)\pi^-(\vec{Q})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle$$
$$+\frac{1}{2}\sum_{\lambda_p=\pm 1/2}\int \frac{\mathrm{d}^3 Q}{(2\pi)^3 2E_{\pi^-}(\vec{Q})2E_p(\vec{Q})}$$
$$\times \frac{1}{E_{\pi^-}(\vec{Q})+E_p(\vec{Q})-E_{\pi^-}(\vec{q})-E_p(\vec{q})-i\varepsilon}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{Q})p(-\vec{Q},\lambda_p)\rangle$$
$$\times \langle p(-\vec{Q},\lambda_p)\pi^-(\vec{Q})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle$$
$$+\frac{1}{2}\sum_{\lambda_n=\pm 1/2}\int \frac{\mathrm{d}^3 Q}{(2\pi)^3 2E_{\pi^0}(\vec{Q})2E_n(\vec{Q})}$$
$$\times \frac{1}{E_{\pi^0}(\vec{Q})+E_n(\vec{Q})-E_{\pi^-}(\vec{k})-E_p(\vec{k})-i\varepsilon}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^0(\vec{Q})n(-\vec{Q},\lambda_n)\rangle$$
$$\times \langle n(-\vec{Q},\lambda_n)\pi^0(\vec{Q})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle$$
$$+\frac{1}{2}\sum_{\lambda_n=\pm 1/2}\int \frac{\mathrm{d}^3 Q}{(2\pi)^3 2E_{\pi^0}(\vec{Q})2E_n(\vec{Q})}$$
$$\times \frac{1}{E_{\pi^0}(\vec{Q})+E_n(\vec{Q})-E_{\pi^-}(\vec{q})-E_p(\vec{q})-i\varepsilon}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^0(\vec{Q})n(-\vec{Q},\lambda_n)\rangle$$
$$\times \langle n(-\vec{Q},\lambda_n)\pi^0(\vec{Q})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle. \quad (B.6)$$

Substituting (B.6) in (B.3) we determine the energy level shift $\epsilon_{1s}^{(2)}$ by the expression

$$\epsilon_{1s}^{(2)} = -\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{\Phi_{1s}^\dagger(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}}$$
$$\times \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{\Phi_{1s}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}}$$
$$\times \Bigg[\frac{1}{2}\sum_{\lambda_p=\pm 1/2}\mathrm{P}\int \frac{\mathrm{d}^3 Q}{(2\pi)^3 2E_{\pi^-}(\vec{Q})2E_p(\vec{Q})}$$
$$\times \frac{1}{E_{\pi^-}(\vec{Q})+E_p(\vec{Q})-E_{\pi^-}(\vec{k})-E_p(\vec{k})}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{Q})p(-\vec{Q},\lambda_p)\rangle$$
$$\times \langle p(-\vec{Q},\lambda_p)\pi^-(\vec{Q})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle$$
$$+\frac{1}{2}\sum_{\lambda_p=\pm 1/2}\mathrm{P}\int \frac{\mathrm{d}^3 Q}{(2\pi)^3 2E_{\pi^-}(\vec{Q})2E_p(\vec{Q})}$$
$$\times \frac{1}{E_{\pi^-}(\vec{Q})+E_p(\vec{Q})-E_{\pi^-}(\vec{q})-E_p(\vec{q})}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{Q})p(-\vec{Q},\lambda_p)\rangle$$
$$\times \langle p(-\vec{Q},\lambda_p)\pi^-(\vec{Q})|\mathcal{L}_{\mathrm{str}}(0)|\pi^-(\vec{q})p(-\vec{q},\sigma)\rangle$$
$$+\frac{1}{2}\sum_{\lambda_n=\pm 1/2}\mathrm{P}\int \frac{\mathrm{d}^3 Q}{(2\pi)^3 2E_{\pi^0}(\vec{Q})2E_n(\vec{Q})}$$
$$\times \frac{1}{E_{\pi^0}(\vec{Q})+E_n(\vec{Q})-E_{\pi^-}(\vec{k})-E_p(\vec{k})}$$
$$\times \langle \pi^-(\vec{k})p(-\vec{k},\sigma)|\mathcal{L}_{\mathrm{str}}(0)|\pi^0(\vec{Q})n(-\vec{Q},\lambda_n)\rangle$$

$$\times \langle n(-\vec{Q}, \lambda_n)\pi^0(\vec{Q})|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q}, \sigma)\rangle$$
$$+\frac{1}{2}\sum_{\lambda_n=\pm 1/2}\mathrm{P}\int\frac{\mathrm{d}^3Q}{(2\pi)^3 2E_{\pi^0}(\vec{Q})2E_n(\vec{Q})}$$
$$\times\frac{1}{E_{\pi^0}(\vec{Q})+E_n(\vec{Q})-E_{\pi^-}(\vec{q})-E_p(\vec{q})}$$
$$\times\langle\pi^-(\vec{k})p(-\vec{k}, \sigma)|\mathcal{L}_{\text{str}}(0)|\pi^0(\vec{Q})n(-\vec{Q}, \lambda_n)\rangle$$
$$\times\langle n(-\vec{Q}, \lambda_n)\pi^0(\vec{Q})|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q}, \sigma)\rangle\Bigg], \quad \text{(B.7)}$$

where P stands for the calculation of the principal value of the integral over $\vec{Q}$.

The energy level width $\Gamma_{1s}$ is given by

$$\Gamma_{1s} = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\Phi_{1s}^\dagger(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}}$$
$$\times\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\Phi_{1s}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}}$$
$$\times\Bigg[\sum_{\lambda_p=\pm 1/2}\int\frac{\mathrm{d}^3Q}{(2\pi)^3 2E_{\pi^-}(\vec{Q})2E_p(\vec{Q})}$$
$$\times\pi\,\delta(E_{\pi^-}(\vec{Q})+E_p(\vec{Q})-E_{\pi^-}(\vec{k})-E_p(\vec{k}))$$
$$\times\langle\pi^-(\vec{k})p(-\vec{k}, \sigma)|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{Q})p(-\vec{Q}, \lambda_p)\rangle$$
$$\times\langle p(-\vec{Q}, \lambda_p)\pi^-(\vec{Q})|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q}, \sigma)\rangle$$
$$+\sum_{\lambda_p=\pm 1/2}\int\frac{\mathrm{d}^3Q}{(2\pi)^3 2E_{\pi^-}(\vec{Q})2E_p(\vec{Q})}$$
$$\times\pi\,\delta(E_{\pi^-}(\vec{Q})+E_p(\vec{Q})-E_{\pi^-}(\vec{q})-E_p(\vec{q}))$$
$$\times\langle\pi^-(\vec{k})p(-\vec{k}, \sigma)|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{Q})p(-\vec{Q}, \lambda_p)\rangle$$
$$\times\langle p(-\vec{Q}, \lambda_p)\pi^-(\vec{Q})|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q}, \sigma)\rangle$$
$$+\sum_{\lambda_n=\pm 1/2}\int\frac{\mathrm{d}^3Q}{(2\pi)^3 2E_{\pi^0}(\vec{Q})2E_n(\vec{Q})}$$
$$\times\pi\,\delta(E_{\pi^0}(\vec{Q})+E_n(\vec{Q})-E_{\pi^-}(\vec{k})-E_p(\vec{k}))$$
$$\times\langle\pi^-(\vec{k})p(-\vec{k}, \sigma)|\mathcal{L}_{\text{str}}(0)|\pi^0(\vec{Q})n(-\vec{Q}, \lambda_n)\rangle$$
$$\times\langle n(-\vec{Q}, \lambda_n)\pi^0(\vec{Q})|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q}, \sigma)\rangle$$
$$+\sum_{\lambda_p=\pm 1/2}\int\frac{\mathrm{d}^3Q}{(2\pi)^3 2E_{\pi^0}(\vec{Q})2E_n(\vec{Q})}$$
$$\times\pi\,\delta(E_{\pi^0}(\vec{Q})+E_n(\vec{Q})-E_{\pi^-}(\vec{q})-E_p(\vec{q}))$$
$$\times\langle\pi^-(\vec{k})p(-\vec{k}, \sigma)|\mathcal{L}_{\text{str}}(0)|\pi^0(\vec{Q})n(-\vec{Q}, \lambda_n)\rangle$$
$$\times\langle n(-\vec{Q}, \lambda_n)\pi^0(\vec{Q})|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q}, \sigma)\rangle\Bigg]. \quad \text{(B.8)}$$

The integrands of the integrals over $\vec{Q}$ should be calculated in the limit $k, q \to 0$. In this limit the matrix elements of the operator $\mathcal{L}_{\text{str}}(0)$ can be approximated by the $S$-wave

scattering lengths as

$$\langle\pi^-(\vec{k})p(-\vec{k}, \sigma)|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{Q})p(-\vec{Q}, \lambda_p)\rangle$$
$$\times\langle p(-\vec{Q}, \lambda_p)\pi^-(\vec{Q})|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q}, \sigma)\rangle =$$
$$\left[\frac{8\pi}{3}\left(m_{\pi^-}+m_p\right)\left(2a_0^{1/2}+a_0^{3/2}\right)\right]^2\delta_{\lambda_p\sigma},$$
$$\langle\pi^-(\vec{k})p(-\vec{k}, \sigma)|\mathcal{L}_{\text{str}}(0)|\pi^0(\vec{Q})n(-\vec{Q}, \lambda_n)\rangle$$
$$\times\langle n(-\vec{Q}, \lambda_n)\pi^0(\vec{Q})|\mathcal{L}_{\text{str}}(0)|\pi^-(\vec{q})p(-\vec{q}, \sigma)\rangle =$$
$$2\left[\frac{8\pi}{3}\left(m_{\pi^-}+m_p\right)\left(a_0^{1/2}-a_0^{3/2}\right)\right]^2\delta_{\lambda_n\sigma}. \quad \text{(B.9)}$$

This yields

$$\epsilon_{1s}^{(2)} = -\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\Phi_{1s}^\dagger(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}}$$
$$\times\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\Phi_{1s}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}}$$
$$\times\Bigg\{\left[\frac{8\pi}{3}\left(m_{\pi^-}+m_p\right)\left(2a_0^{1/2}+a_0^{3/2}\right)\right]^2$$
$$\times\mathrm{P}\int\frac{\mathrm{d}^3Q}{(2\pi)^3 2E_{\pi^-}(\vec{Q})2E_p(\vec{Q})}$$
$$\times\frac{1}{E_{\pi^-}(\vec{Q})+E_p(\vec{Q})-m_{\pi^-}-m_p}$$
$$+2\left[\frac{8\pi}{3}\left(m_{\pi^-}+m_p\right)\left(a_0^{1/2}-a_0^{3/2}\right)\right]^2$$
$$\times\mathrm{P}\int\frac{\mathrm{d}^3Q}{(2\pi)^3 2E_{\pi^0}(\vec{Q})2E_p(\vec{Q})}$$
$$\times\frac{1}{E_{\pi^0}(\vec{Q})+E_n(\vec{Q})-m_{\pi^-}-m_p}\Bigg\} \quad \text{(B.10)}$$

and

$$\Gamma_{1s} = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\Phi_{1s}^\dagger(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}}$$
$$\times\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\Phi_{1s}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}}$$
$$\times\Bigg\{2\left[\frac{8\pi}{3}\left(m_{\pi^-}+m_p\right)\left(2a_0^{1/2}+a_0^{3/2}\right)\right]^2$$
$$\times\int\frac{\mathrm{d}^3Q}{(2\pi)^3 2E_{\pi^-}(\vec{Q})2E_p(\vec{Q})}$$
$$\times\pi\,\delta(E_{\pi^-}(\vec{Q})+E_p(\vec{Q})-m_{\pi^-}-m_p)$$
$$+4\left[\frac{8\pi}{3}\left(m_{\pi^-}+m_p\right)\left(a_0^{1/2}-a_0^{3/2}\right)\right]^2$$
$$\times\int\frac{\mathrm{d}^3Q}{(2\pi)^3 2E_{\pi^0}(\vec{Q})2E_n(\vec{Q})}$$
$$\times\pi\,\delta(E_{\pi^0}(\vec{Q})+E_n(\vec{Q})-m_{\pi^-}-m_p)\Bigg\}. \quad \text{(B.11)}$$

The subsequent calculation of $\epsilon_{1s}^{(2)}$ and $\Gamma_{1s}$ we discuss in sect. 5.

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